Involution on quiver varieties and quantum symmetric pairs

Hiraku Nakajima, Kavli IPMU, University of Tokyo

合流する組み合わせ論と表現論

Confluence of combinatorics and representation theory

2025-10-22 @ RIMS

Part I: σ-quiver variety

$Q=(Q_0,Q_1)$: quiver of type ADE

$*$ = diagram automorphism given by w_0 = longest element of the Weyl group

$$w_0(\alpha_i) = -\alpha_{i^*}$$

e.g. $A_{\ell-1}$ 

V, W : Q_0-graded vector spaces

$v, w \in \mathbb{Z}_{\geq 0}^{Q_0}$ dimension vectors $\quad \lambda = \sum w_i \varpi_i \ , \ \mu = \lambda - \sum v_i \alpha_i$ weights

$\mathfrak{M}_\theta(\lambda,\mu)$ = quiver variety with stability parameter θ $\quad \{(B,a,b) \mid \text{moment map},\ \theta\text{-stable}\}/G_V$

Consider a sequence of isomorphisms

$$\mathfrak{M}_\theta(\lambda,\mu) \xrightarrow{t} \mathfrak{M}_{-\theta}(\lambda,\mu) \xrightarrow{*} \mathfrak{M}_{\underbrace{-\theta^*}_{= w_0\theta}}(\lambda^*,\mu^*) \xrightarrow{S_{w_0}} \mathfrak{M}_\theta(\lambda^*, \underbrace{w_0\mu^*}_{= -\mu})$$

take transposes of linear maps $\qquad$ diagram auto $\qquad$ reflection functor

<u>Remark</u>: we can also compose a diagram automorphism σ' (split vs quasisplit symmetric pair)

— Assume $\lambda^* = \lambda$, $\mu = 0$ (more generally $\sigma'(\lambda^*) = \lambda$, $-\sigma'(\mu) = \mu$)

$\mathfrak{M}_\theta(\lambda, \mu=0) \longrightarrow \mathfrak{M}_\theta(\lambda^* = \lambda, -\mu = 0)$ $\sim$ same as the original space

Since quiver varieties <u>depend</u> on W : vector space, rather than λ,
we need to choose $W \cong W^*$ Q_0-grading is twisted by $*$

— Assume $\cong$ is given by $(+)$: orthogonal form or $(-)$: symplectic form

Then we have an involution $\mathfrak{M}_\theta(\lambda, \mu) \circlearrowleft \sigma$

<u>Def</u> [Y. Li] Assume θ : generic ($\Rightarrow \mathfrak{M}_\theta(\lambda, \mu)$: smooth)

$\mathfrak{M}_\theta(\lambda, \mu)^\sigma = \sigma$-quiver variety

<u>Ex</u> 1. $\sigma' = \mathrm{id}$ $\Rightarrow$ moduli space of SO/Sp instantons on ALE spaces

<u>Ex</u> 2 type A frame $\sigma' \neq \mathrm{id}$ only at 1 $\Longrightarrow$ cotangent bdles of flags $(\mathbb{C}^N \supset V_1 \supset V_2 \supset \cdots \supset V_2^\perp \supset V_1^\perp \supset \{0\})$ $\perp$

Let me concentrate on the following example:

type $A_{\ell-1}$

$$\begin{tikzcd} n \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & n \arrow[d, dash] \\ \boxed{n} & & \boxed{n} \end{tikzcd}$$

$\lambda = n(\varpi_1 + \varpi_{\ell-1})$

$\mu = 0$

$\mathfrak{M}^\sigma(n) \equiv \mathfrak{M}_0(\lambda, \mu)^\sigma$

Put orthogonal or symplectic on $\mathbb{C}^n \oplus \mathbb{C}^n$

$+ : SO$ $\quad - : Sp$

$g \in GL_n \curvearrowright \mathfrak{M}^\sigma$ mult. by g mult. by g^{*-1}

Consider $\rho : \mathbb{C}^\times \to GL_n$

$$\rho: t \longmapsto \begin{bmatrix} \overbrace{t \ \ddots \ t}^{n_1} & \\ & \overbrace{1 \ \ddots \ 1}^{n_2} \end{bmatrix}$$

$n_1 + n_2 = n$

Consider $(\mathfrak{M}^\sigma)^{\rho(\mathbb{C}^*)}$ fixed point locus.

$(\text{usual quiver variety})^{\rho(\mathbb{C}^*)} = $ product of quiver varieties

Claim. $(\mathfrak{M}^\sigma(n))^{\rho(\mathbb{C}^*)} = \bigsqcup_{v_1, \dots, v_\ell} \mathfrak{M}^\sigma(n_2) \times \mathfrak{M}\left(\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & v_\ell \\ \boxed{n_1} \end{tikzcd}\right)$

σ-quiver variety ordinary quiver variety

$$\cong^\sigma \ \mathfrak{M}\left(\begin{tikzcd} n_1 - v_1 \arrow[r, dash] & \cdots \arrow[r, dash] & n_1 - v_\ell \arrow[d, dash] \\ & & \boxed{n_1} \end{tikzcd}\right)$$

Basically because $Z_{SO/Sp_{2n}}(\rho(t)) = SO/Sp_{2n_2} \times GL_{n_1}$ vs $Z_{GL_n}(\rho(t)) = GL_{n_1} \times GL_{n_2}$

Maulik-Okounkov stable envelope :

$$H^*_{equ.}\left((\mathcal{M}^\sigma(n))^{\rho(\mathbb{C}^*)}\right) \longrightarrow H^*_{equ}(\mathcal{M}^\sigma(n))$$

characterized by several properties

becomes isomorphism over $\mathrm{Frac}\, H^*_{equ}(pt)$

$$\begin{gathered} \| \\ \bigoplus H^*_{equ}(\mathcal{M}^\sigma(n_2)) \otimes H^*_{equ}\Big(\mathcal{M}\big(\underset{\boxed{n_1}}{\overset{|}{(v_1)}} \cdots (v_g)\big)\Big) \end{gathered}$$

For ordinary quiver varieties, stable envelope for ρ and for ρ^{-1} gives a geometric construction of R-matrix

$$F_1[u_1] \otimes F_2[u_2] \circlearrowleft R(u_1 - u_2) \quad \text{(rational isom.)}$$

In our case, for $\mathcal{H} = H^*_{equ}(\mathcal{M}^\sigma(n_2))$

$$\mathcal{H} \otimes F[u] \circlearrowleft K(u) \quad \underline{\text{K-matrix}}$$

σ-quiver variety

ordinary quiver variety

This is the observation by Y. Li.

(We later consider the case $\mathcal{H}$ = base ring)

Remark $T^n \subset GL_n$ $\qquad$ $\mathcal{M}^\sigma(n)^{T_n}$: finite pts

$\Rightarrow$ K-matrix can be computed explicitly.

$\Rightarrow$ Geometric application :
One can compute Betti numbers

$\mathcal{M}^\sigma(n)$: disconnected for $(+)$-type SO
connected for $(-)$-type Sp

Part II Twisted Yangian

§1. definition (very rough)

$\mathfrak{g}$ = semisimple Lie alg $/\mathbb{C}$ e.g. $\mathfrak{sl}_n$

σ = involution

$(\mathfrak{g}, \mathfrak{g}^\sigma)$: symmetric pair $(\mathfrak{sl}_n, \mathfrak{so}_n)$, $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$, $(\mathfrak{sl}_n, \mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q))$

not quasi split

$n = p+q$

quasi split only if $p=q$, $|p-q|=1$

$U_q(\mathfrak{g})$: quantum enveloping alg Hopf alg Δ = coproduct

$\not\supset U_q(\mathfrak{g}^\sigma)$ no natural embedding

but $U_q^{tw}(\mathfrak{g}^\sigma) \subset U_q(\mathfrak{g})$ (quantum symmetric pair)

s.t.
- different from $U_q(\mathfrak{g}^\sigma)$
- $\left(U_q^{tw}(\mathfrak{g}^\sigma) \subset U_q(\mathfrak{g})\right)\Big|_{q=1} = U(\mathfrak{g}^\sigma) \subset U(\mathfrak{g})$
- $\Delta U_q^{tw}(\mathfrak{g}^\sigma) \subset U_q(\mathfrak{g}) \otimes U_q^{tw}(\mathfrak{g}^\sigma)$

coideal property

....., Noumi, ...

- ∞-dimensional version

  $Y = Y(\mathfrak{g})$: Yangian of $\mathfrak{g}$: quantization of $U(\mathfrak{g}[z])$

  filtered alg. s.t. associated graded

  rational version (param. $\hbar = \log q$)

  $Y^{tw}(\mathfrak{g}^\sigma) \subset Y(\mathfrak{g})$

  - different from $Y(\mathfrak{g}^\sigma)$
  - quantization of $U(\mathfrak{g}[z]^\sigma) \subset U(\mathfrak{g}[z])$

    involution of $\mathfrak{g}$

    $\mathfrak{g}[z] \circlearrowleft \sigma = \sigma \otimes (-1)$
  - $\Delta Y^{tw}(\mathfrak{g}^\sigma) \subset Y(\mathfrak{g}) \otimes Y^{tw}(\mathfrak{g}^\sigma)$

$$\begin{array}{ccccc} \mathrm{Rep}\, Y(\mathfrak{g}) & \times & \mathrm{Rep}\, Y^{tw}(\mathfrak{g}^\sigma) & \longrightarrow & \mathrm{Rep}\, Y^{tw}(\mathfrak{g}^\sigma) \\ V & & W & \longmapsto & V \otimes W \end{array}$$

module category over monoidal category

<u>Remark</u> σ-quiver variety $\mathfrak{M}_\theta(\lambda, \mu)^\sigma$: smooth

$\Rightarrow$ σ on $\mathfrak{g}$ is quasi-split type. i.e. $\begin{pmatrix}\text{diagram}\\ \text{auto}\end{pmatrix} \circ \sigma_{\text{Chevalley}}$

<u>Conj.</u> Allow singularity on $\mathfrak{M}_\theta(\lambda, \mu)^\sigma$ for more general σ

§2. Reflection equation

Q. How to construct $Y^{tw}(\mathfrak{g}^\sigma) \subset Y(\mathfrak{g})$?

A1. Drinfeld original presentation $Y(\mathfrak{g}) = \langle \mathfrak{g}, \overbrace{J\mathfrak{g}}^{\text{analog of } \mathfrak{g}\cdot z} \rangle / \text{relation}$

Belliard–Regelskis $Y^{tw}(\mathfrak{g}^\sigma) = \langle \mathfrak{g}^\sigma, J\mathfrak{f} \rangle$ $\quad \mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{f}$ $\quad$ (Assume $\sigma \neq id$)

A2. From K-matrix (analog of R-matrix)

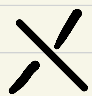
<u>Recall</u> RTT construction

representation of Yangian

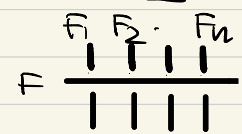
$F_1[u] \otimes F_2[v] \circlearrowleft R(u-v)$ intertwiner between Δ and Δ^{op}

spectral parameter

= equivariant variable



rational in $u-v$



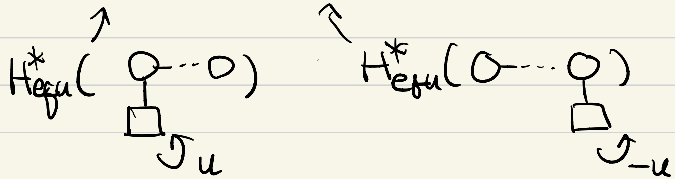
∘ monodromy matrix $F[u] \otimes F_1[u_1] \otimes \cdots \otimes F_n[u_n] \circlearrowleft R_{FF_n}(u-u_n) \cdots R_{FF_1}(u-u_1) =: T_F(u)$

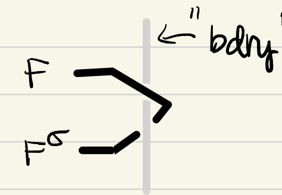
$X \overset{\text{def.}}{=} \langle$ matrix elements of $T_F(u)$ expanded at $u=\infty \rangle$

extended Yangian $\subset \prod \mathrm{End}_{\mathbb{C}[u_1,\cdots,u_n]}(F_1[u_1] \otimes \cdots \otimes F_n[u_n])$

Yang–Baxter equation $\Rightarrow$ RTT relation

K-matrix $K(u) : F[u] \longrightarrow F^\sigma[-u]$



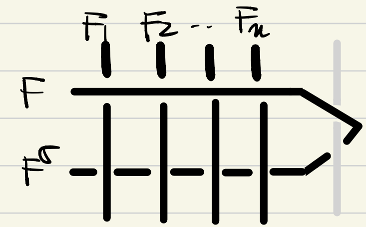


(Assume $\mathcal{H}$ = base ring)

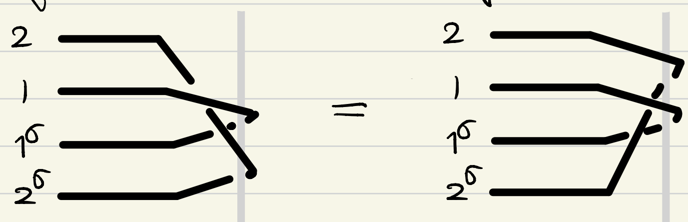
analog of monodromy matrix

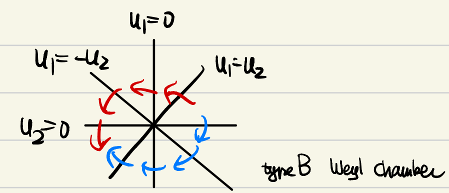
$$S_F(u) := R_{F_1 F^\sigma}(u+u_1)_{21} \cdots R_{F_n F^\sigma}(u+u_n)_{21} K_F(u) R_{FF_n}(u-u_n) \cdots R_{FF_1}(u-u_1)$$

$$: F[u]\otimes F_1[u_1] \cdots \otimes F_n[u_n] \to F^\sigma[-u]\otimes F_1[u_1]\otimes \cdots \otimes F_n[u_n]$$



analog of YB equation = reflection equation





$$K_2(u_2) R_{21^\sigma}(u_1+u_2)_{21} K_1(u_1) R_{12}(u_1-u_2) = R_{2^\sigma 1^\sigma}(u_1-u_2)_{21} K_1(u_1) R_{12^\sigma}(u_1+u_2) K_2(u_2)$$

X^{tw} (extended twisted Yangian) $=$ coefficients of matrix elements (w.r.t. F and F^σ)
$\subset \prod \mathrm{End}_{\mathbb{C}[u_1,\cdots,u_n]}(F_1[u_1]\otimes\cdots\otimes F_n[u_n])$ expanded at $u\to\infty$

<u>Th</u> $H^*_{equ}(M^\sigma(n))$ is a representation of X^{tw} for the pair $(\mathfrak{g},\mathfrak{g}^\sigma)=(\mathfrak{sl}_\ell,\mathfrak{so}_\ell)$
for $(+)$ type.
For $(-)$ type, compatibility of polarization is not satisfied.
$\Rightarrow$ reflection equation does not hold.

<u>Rem</u> cotangent of flags case : $X^{tw}=$ Molev–Ragoucy reflection equation alg.
$X^{tw}\subset X$ is different for $+/-$ type